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CHARGED-PARTICLE MOTION IN MULTIDIMENSIONAL MAGNETIC-FIELD TURBULENCE[1]

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ABSTRACT

We present a new analysis of the fundamental physics of charged-particle motion in a turbulent magnetic field using a numerical simulation. The magnetic field fluctuations are taken to be static and to have a power spectrum which is Kolmogorov. The charged particles are treated as test particles. It is shown that when the field turbulence is independent of one coordinate (i.e., $\boldsymbol{k}$ lies in a plane), the motion of these particles across the magnetic field is essentially zero, as required by theory. Consequently, the only motion across the average magnetic field direction that is allowed is that due to field-line random walk. On the other hand, when a fully three-dimensional realization of the turbulence is considered, the particles readily cross the field. Transport coefficients both along and across the ambient magnetic field are computed. This scheme provides a direct computation of the Fokker-Planck coefficients based on the motions of individual particles, and allows for comparison with analytic theory.

Subject headings: cosmic rays — diffusion — magnetic fields — turbulence

1. INTRODUCTION

A large number of investigations of charged-particle motion in a fluctuating magnetic field were published in the mid 1960s to mid 1970s. Jokipii (1966) explicitly related the spatial transport of charged particles to magnetic-field turbulence (Batchelor 1960). There followed much related work primarily concerned with the transport of cosmic rays in interstellar and interplanetary space (e.g., Hasselman & Wibberenz 1968; Roelof 1968; Jokipii 1971; Goldstein, Klimas, & Sandri 1975; Earl 1974; Wentzel 1974; Luhmann 1976). These analyses were based primarily on quasi-linear theory. Subsequently, a number of authors incorporated nonlinear effects of the magnetic fluctuations on the lowest order particle orbits (e.g., Jones, Kaisser, & Birmingham 1973). For a review of these see Völk (1968).

Recently, it was demonstrated analytically (Jokipii, Kóta, & Giacalone 1993) that when at least one ignorable spatial coordinate exists in the description of the electromagnetic field, the motion of particles across the magnetic-field lines is artificially suppressed. This explains why, in those previous analyses in which only one- or two-dimensional fluctuations were considered (i.e., the magnetic-field vector depends on only one or two spatial coordinates), the resulting cross-field diffusion coefficient was only that due to field-line mixing (e.g., Jokipii 1966; Forman, Jokipii, & Owens 1974), since the particles are tied to the same field line on which they started. This new insight has prompted us to revisit the general subject of transport in turbulent magnetic fields.

The analysis of Jokipii et al. (1993) concludes that motion across field lines is not possible when the magnetic field depends on less than three spatial coordinates. However, their analysis did *not* show that cross-field transport actually *does* occur when a three-dimensional field realization is considered. In this report we demonstrate that motion normal to the field lines *does* occur in three-dimensional turbulence and is suppressed in similar two-dimensional turbulence (and, by inference, one-dimensional as well). Therefore, is is vital to consider a fully three-dimensional description of the magnetic field to elucidate the full characteristics of charged-particle motion in such fields. The results in this paper are based on only a few simulation results and consequently only a few general conclusions can be drawn from this study. In a future investigation a more comprehensive analysis will be given.

We consider a particle of mass m and charge q moving in a static magnetic field $\boldsymbol{B}(\boldsymbol{r}) = mc\boldsymbol{\Omega}(\boldsymbol{r})/q$, where c is the speed of light. The trajectories are traced by solving each particle's equation of motion:

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{u} \times \boldsymbol{\Omega} \tag{1}$$

where $\boldsymbol{u}$ is the particle velocity. A number of particles are released from the origin in a given magnetic-field realization, with a speed u_0, but in a random direction.

2. THE MAGNETIC FIELD

The field $\boldsymbol{\Omega}$ in equation (1) may be written $\boldsymbol{\Omega}(\boldsymbol{r}) = \boldsymbol{\Omega}_0 + \delta\boldsymbol{\Omega}(\boldsymbol{r})$, where $\boldsymbol{\Omega}_0 = q\boldsymbol{B}_0/mc$ is the background field, and $\delta\boldsymbol{\Omega}$ is the fluctuation about the mean. Note that we do not require $\delta\boldsymbol{\Omega}$ to be small. A three-dimensional realization of $\boldsymbol{\Omega}(\boldsymbol{r})$ is obtained by summing over a given number of transverse waves, with random polarization, each with a $\boldsymbol{k}$ that is oriented in a random direction defined by two angles θ and ϕ. A form which satisfies $\nabla \cdot \boldsymbol{B} = 0$ may be written

$$\delta\boldsymbol{\Omega}(\boldsymbol{r}) = \sum_{k} \Omega(k)[\cos \alpha(k)\hat{y}' \pm i \sin \alpha(k)\hat{z}'] \exp [ikx' + i\beta(k)] \tag{2}$$

where $\alpha(k)$ and $\beta(k)$ are random numbers between 0 and 2π. The $\Omega(k)$ are chosen to give the desired power spectrum of the magnetic field. The primed system is related to the unprimed system by the rotation

$$\boldsymbol{r}'_{3D} = \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & \sin\theta \\ -\sin\phi & \cos\phi & 0 \\ -\sin\theta\cos\phi & -\sin\theta\sin\phi & \cos\theta \end{pmatrix} \boldsymbol{r}_{3D}\,, \tag{3}$$

where θ and ϕ are functions of k.

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For any given realization, and for each k, there are five random numbers: $0 < \theta(k) < \pi$, $0 < \phi(k) < 2\pi$, $0 < \beta(k) < 2\pi$, $0 < \alpha(k) < 2\pi$, and the plus or minus.

A two-dimensional realization which satisfies $\nabla \cdot \boldsymbol{B} = 0$ is also given by equation (2), except the relationship between the primed and unprimed systems is given by the two-dimensional rotation obtained from equation (3) with $\phi = 0$. In this case, only four sets of random numbers need to be chosen (α, β, θ, and $\pm$).

The irregular magnetic field is taken to be generated by Kolmogorov turbulence.

$$\Omega(k) = \Omega(k_{\min})\left(\frac{k}{k_{\min}}\right)^{-\gamma/2}, \qquad (4)$$

where $k_{\min}$ corresponds to the longest wavelength and $\gamma = 5/3$ is the spectral index of the power spectrum $\Omega^2(k)$. $\Omega(k_{\min})$ is obtained from the total energy density which is defined in terms of the magnetic field, $B(k) = mc\Omega(k)/e$ as

$$S = \sum_k \frac{B^2(k)}{8\pi} = \frac{m^2c^2}{8\pi q^2}\,\Omega^2(k_{\min}) \sum_k \left(\frac{k}{k_{\min}}\right)^{-\gamma}. \qquad (5)$$

In the present case, 50 values of k were used, evenly spaced logarithmically, with wavelengths between $\frac{1}{5}$ and $10u_0/\Omega_0$, where Ω_0 is the proton gyrofrequency in the background magnetic field B_0. The total energy density was set to $B_0^2/8\pi$ (i.e., there is as much energy density in the fluctuations as there is in the background field).

The magnetic field $\boldsymbol{\Omega}(\boldsymbol{r})$ is generated at every time step, at each particle position. Note that the same realization (set of random numbers) is used for each particle; i.e., only $\boldsymbol{r}$ changes. This procedure is quite time consuming; however, the alternative of generating the magnetic field at the beginning for the given value is worse, since the volume must be so large. An estimate of the required volume is $V \sim 100\lambda_{\max}^3$, where $\lambda_{\max}$ is the longest wavelength in the fluctuations and the factor of 100 is an estimate of the coherence length along the mean field direction. Moreover, the distance between lattice points upon which an interpolation would be performed is $0.25\lambda_{\min}$, where $\lambda_{\min}$ is the smallest wavelength. Therefore, there would need to be $6400(\lambda_{\max}/\lambda_{\min})^3$ computations performed at the start of the simulation. For $\lambda_{\max} = 50\lambda_{\min}$, this gives 8×10^8, which is not only at least as time consuming as generating the turbulence at each point along the particle trajectory, but is also unfeasible in terms of computer memory storage. Alternate ways of circumventing these problems will be pursued in the future.

3. RESULTS AND DISCUSSION

Sample particle trajectories computed in the generated field are shown in Figure 1. The trajectories are projected onto the x-z plane, and the average field points in the z-direction. In the upper panel, the trajectory of a particle using the two-dimensional turbulence is shown, whereas in the bottom panel, the trajectory of the same particle using three-dimensional turbulence is shown. In this example, each particle was followed for a time $5000\Omega_0^{-1}$. It is clear that in the case of two-dimensional turbulence, the particle essentially traces out a magnetic field line. On the other hand, when the more realistic three-dimensional case is considered the particle diffuses in both directions. Projections onto the x-z plane of field lines shown in Figure 2 for reference (note that they are shifted to the right for visibility).

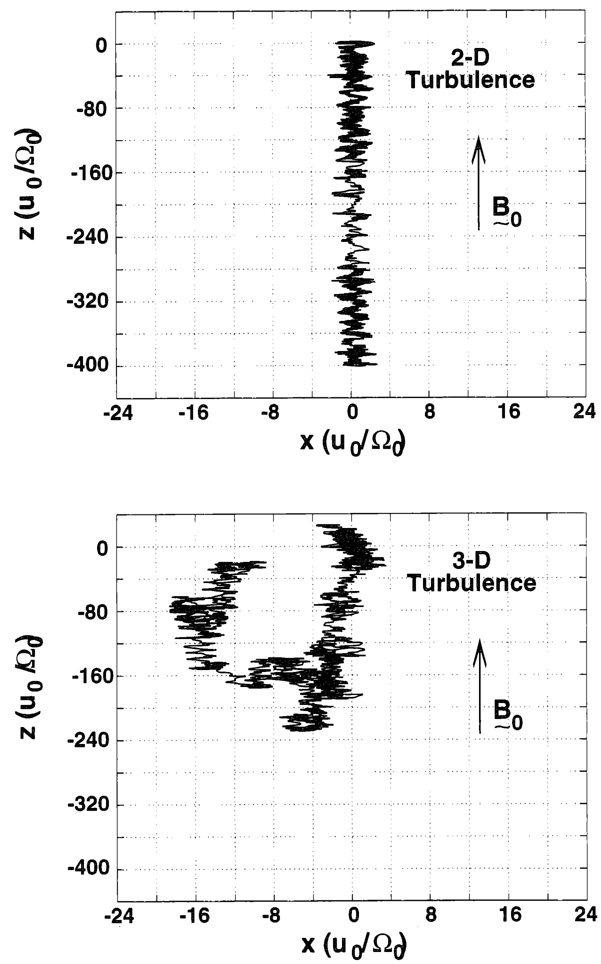

FIG. 1.—Sample particle trajectories projected onto the x-z plane. The initial speed of the particle (the same in each case) is u_0 and has a gyroradius u_0/Ω_0 (one unit on this scale).

To illustrate that this is a general feature of the dimensionality effect, a scatter plot of the final x and z locations of 3000 test particles (injected at the origin) is shown in Figure 2. The time is $1000\Omega_0^{-1}$. Note the difference in length scales indicating that in the three-dimensional turbulence case, as one expects, the particles diffuse farther along the field than across. It is also clear from Figure 2 that in the two-dimensional case, the particles do not stray a distance of more than about one gyroradius normal to the original field line.

Further evidence that particles are being transported across the field in three-dimensions but not two dimensions is shown in Figure 3. Shown here are distributions of absolute value of the x-displacement from the origin of the 3000 test particles shown in Figure 2, at two successive times: $250\Omega_0^{-1}$ (solid histograms) and $1000\Omega_0^{-1}$ (dotted lines). In the case of two-dimensional turbulence (upper panel) there is clearly little difference between the distributions at two different times, indicating that there is no diffusion taking place. The actual spread is strictly due to field-line mixing. Since we have a finite interval in k over which the fluctuations are generated, there is no net diffusion of the field lines across the ambient magnetic field since there is no power in the mode corresponding to $k = 0$ (Jokipii 1966; Forman et al. 1974). For the three-dimensional case (lower panel), there is significant diffusion

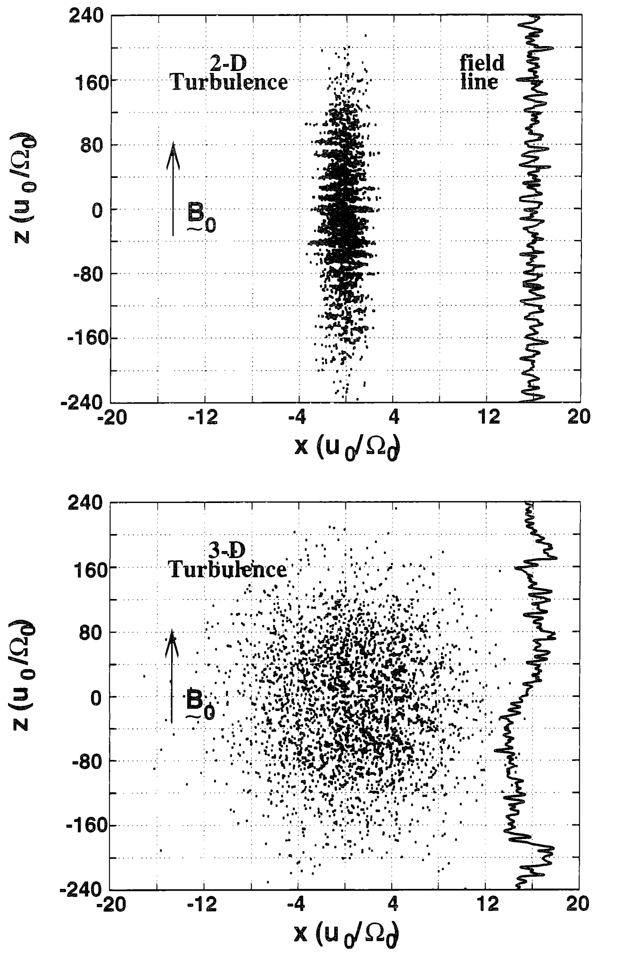

FIG. 2.—Scatter plot of the final ($t = 1000\Omega_0^{-1}$) x-z locations of 3000 test particles released isotropically with a speed of u_0 at the origin. The gyroradius of the particles is u_0/Ω_0. Projections into the x-z plane of the field lines passing through the origin are shown for reference (note that they have been shifted to the right for visibility).

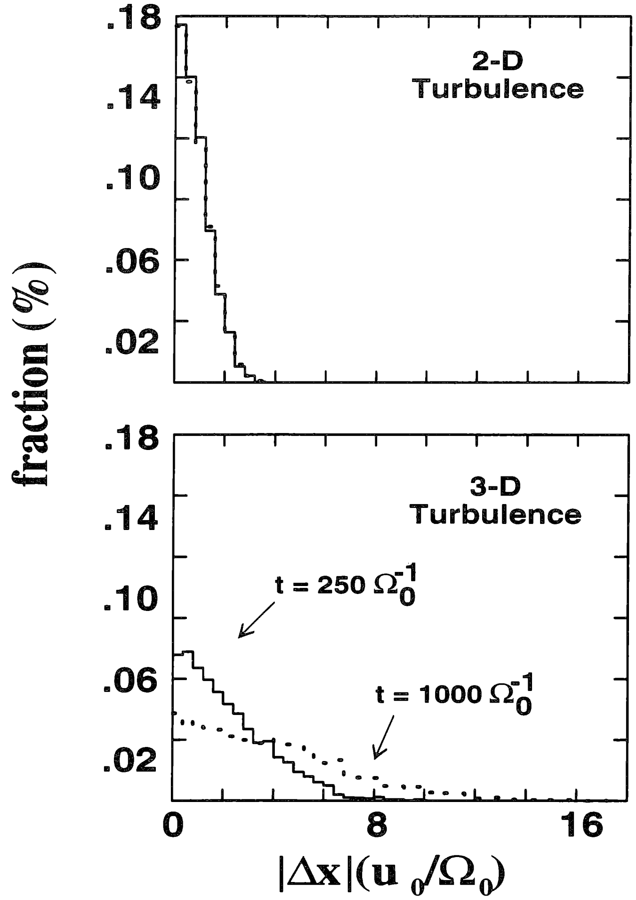

FIG. 3.—Distributions of the absolute value of the distance along the x-direction from the origin of the test particles displayed in Fig. 2. Distributions at two separate times are shown.

taking place as illustrated by the broader distribution at the later time.

The diffusion tensor is defined in terms of the Fokker-Planck coefficients given by

$$\kappa_{ij} = \frac{\langle \Delta r_i \Delta r_j \rangle}{2\,\Delta t} , \tag{6}$$

where $\Delta \boldsymbol{r}$ is the displacement of a particle from its original position after a time Δt. The angle brackets denote ensemble averages, which can also be viewed as averages over many particle trajectories in a single field realization. The definition reads

$$\langle \Delta r_i \Delta r_j \rangle = \frac{1}{N} \sum_{i=1}^{N} \Delta r_i \Delta r_j , \tag{7}$$

where N is the number of particles used in the average.

In equation (7), because of the central limit theorem, $\langle \Delta r_i \Delta r_j \rangle$ should exhibit a linear increase with time for times longer than a coherence time. On the other hand, if there is no scattering and the ambient field points in the z-direction, then $(\Delta z)^2 \propto t^2$. This is the initial behavior of a collection of particles even in the presence of turbulence and will only persist until the coherence time of the velocity has been reached.

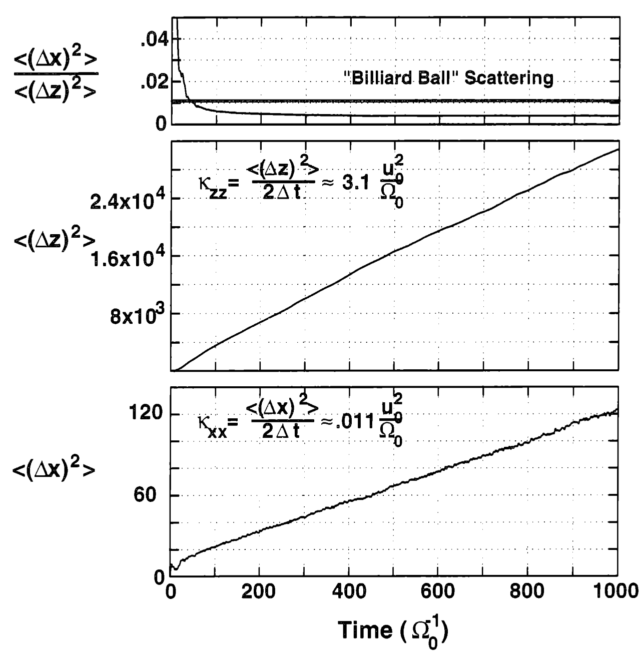

FIG. 4.—Plots of $\langle(\Delta x)^2\rangle$ (across the ambient field) and $\langle(\Delta z)^2\rangle$ (along the ambient field) as functions of time. The insets depict the diffusion coefficients computed from the slopes of these lines. The upper panel displays their ratio.

In Figure 4, two components of $\langle \Delta r_i \Delta r_j \rangle$ are plotted, based on the motions of 3000 test particles. The curves show that $\langle (\Delta r)^2 \rangle$ increases roughly linearly with time after $\sim 100\Omega_0^{-1}$. The inserts display the computed diffusion coefficients both along (middle panel) and across (lower panel) the ambient magnetic-field direction and are simply the slopes of a least-squares fit to the data. Plotted in the upper panel, is the ratio of the cross-field to parallel diffusion coefficients which is roughly constant after $t = 100\Omega_0^{-1}$. Also shown is the standard "billiard-ball" scattering result

$$\frac{\kappa_\perp}{\kappa_\parallel} = \frac{1}{1 + (\lambda_\parallel / r_g)^2}, \quad (8)$$

where $\lambda_\parallel = 3\kappa_\parallel \Omega_0 / r_g$ is the parallel mean free path, and $r_g = u_0/\Omega_0$ is the gyroradius. Taking the ratio of the slope of the $\langle (\Delta x)^2 \rangle$ line to the $\langle (\Delta z)^2 \rangle$ line gives 3.55×10^{-3}. Using the simulated parallel diffusion coefficient (see insert to the middle panel of Fig. 4) to get the parallel mean free path ($\lambda_\parallel \simeq 9.3 r_g$) and inserting it into equation (8) gives a ratio of the perpendicular to parallel diffusion coefficients of 1.14×10^{-2}, which indicates that the simple hard sphere scattering picture is in reasonable agreement with the simulations. However, it gives a value that is about a factor of 3 larger than the simple picture. We do not fully understand this discrepancy at present, although it may be related to the difficulty of scattering through 90° pich angle. Note that the simulated parallel diffusion coefficient in the present case is likely to be larger than if we had used a larger range of k's since there is no power in modes corresponding to wavelengths less than $\frac{1}{5} u_0/\Omega_0$. If more power were to exist in the small-wavelength regime, the scattering of particles with pitch angles near 90° would be more efficient. Moreover, the choice of the number of modes is also of importance since the resonant mode must be represented for efficient scattering to take place. All of these factors may lead to a better agreement between the test particle simulations and the hard sphere scattering picture. We will address these issues in a future analysis.

4. SUMMARY

We have shown that in a magnetic field which fluctuates irregularly with a Kolmogorov like spectrum, in three spatial directions, charged particles diffuse both along and across the magnetic field. On the other hand, when the turbulence is similar, but two-dimensional, cross-field diffusion is due only to field-line mixing, since particles are strictly tied to the same field line on which they started (Jokipii et al. 1993). The technique developed here will be useful in addressing the as yet unsolved problem of cross-field transport in turbulent astrophysical plasmas.

We conclude that the transport of charged particles across a magnetic field is a natural consequence of the interaction of the particles with three-dimensional turbulence and is unphysically suppressed in models using lesser dimensions. This result poses an interesting problem for the large number of analytical models and numerical simulations in which, for convenience (often for computational reasons), the assumption of one or two-dimensionality is used.

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