CHARGED-PARTICLE MOTION IN ELECTROMAGNETIC FIELDS HAVING AT LEAST ONE IGNORABLE SPATIAL COORDINATE.

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ABSTRACT

We give a rigorous derivation of a theorem showing that charged particles in an arbitrary electromagnetic field with at least one ignorable spatial coordinate remain forever tied to a given magnetic field line. Such a situation contrasts with the significant motions normal to the magnetic field that are expected in most real three-dimensional systems. It is pointed out that while the significance of the theorem has not been widely appreciated, it has important consequences for a number of problems and is of particular relevance for the acceleration of cosmic rays by shocks.

Subject headings: acceleration of particles — cosmic rays — methods: analytical — MHD

1. INTRODUCTION.

The problem of particle motion in electromagnetic fields is fundamental and pervades both laboratory and astrophysical plasma physics. In spite of the conceptual simplicity of the problem, theoretical analysis and numerical modeling of the particle motions can be difficult. Often, in order either to simplify the analytical work or to reduce the demand on computer resources, the system of interest is considered in only one or two spatial dimensions with the hope that no essential physics will be lost. It is readily shown that this approximation does have a number of important general consequences that can result in the loss of essential physics, and it must therefore be used with extreme caution. On the other hand, some real physical systems have a symmetry such that one (or even two) coordinates are ignorable. In such cases this imposes a real constraint on the possible motions of charged particles. While we will discuss an example of this type of real symmetry, the primary emphasis of this paper will be to address the profound effect that this reduction of dimensionality has on theoretical studies of particle acceleration.

Jokipii, Kóta, & Giacalone (1993, hereafter JKG) presented a general theorem regarding spatial constraints to charged-particle motion in an electromagnetic field that has at least one ignorable spatial coordinate and discussed explicitly the application of this theorem to hybrid simulations of collisionless shock waves. In the context of hybrid plasma simulations the result has been stated in previous papers (Thomas & Brecht 1988; Thomas & Winske 1991), so it has been known for a while by some workers in the field. In auroral and radiation belt physics it has been known and applied for much longer (Störmer 1955). The essence of this theorem is that in such systems a charged particle is effectively forever tied, in a sense that we will define later, to the same magnetic line of force, except for motion along the ignorable coordinate or for the case of a magnetic field entirely aligned in the ignorable direction. Cross-field motion is expected in many real, three-

dimensional systems (Giacalone & Jokipii 1994). This constraint on particle motion represents a critical loss of physics in many situations in which the ignorability of the coordinate is an artifact of the analysis rather than a property of the real system. Since JKG presented the theorem in a heuristic manner, and, because it appears that its implication has not been always understood or believed, in the present paper we present a rigorous derivation of the theorem and include an expanded discussion of its implications.

The most salient application of the theorem concerns one- and two-dimensional hybrid and full plasma simulations, which have been employed by numerous researchers to study particle acceleration in collisionless shocks. A general conclusion of these investigations is that, while the simulations can generate substantial nonthermal populations of particles in quasi-parallel shock systems, none have been able to demonstrate any significant acceleration of a stochastic nature of ions and electrons in quasi-perpendicular shock environments. Hybrid simulations that model the quasi-parallel portion of the Earth's bow shock (Trattner & Scholer 1991; Scholer, Trattner, & Kucharek 1992) produced nice comparisons with AMPTE suprathermal ion observations, following hard on the heels of the successful modeling of this data by Ellison, Möbius, & Paschmann (1990) using a Monte Carlo technique to describe particle convection and diffusion. In contrast, highly oblique and quasi-perpendicular hybrid simulation models of the acceleration of interstellar pick up ions at the heliospheric termination shock (e.g., Liewer, Goldstein, & Omidi 1993; Kucharek & Scholer 1995; Liewer, Rath, & Goldstein 1995) fail to create any measurable nonthermal ions for field obliquities above around 50°. For full plasma simulations, which have had only very limited application to shock acceleration problems, the situation is similar. It has been concluded by such studies (Gallant et al. 1992) that perpendicular relativistic electron-positron shocks, such as those which would occur in pulsar wind termination shocks, are incapable of accelerating the ambient particles to nonthermal energies. This situation is modified slightly by the inclusion of a proton component (Hoshino et al. 1992), which spawns magnetosonic waves that can provide limited (second-order stochastic) acceleration of positrons only. We suggest that, in light of the following discussion, such results may be viewed as the consequence of the reduced dimensionality and the concomitant loss of cross-field diffusion.

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2. THE THEOREM: ITS SPECIFICATION AND MEANING

The theorem states that for a system of electromagnetic fields that have at least one ignorable coordinate, the corresponding component of canonical momentum is conserved. The conservation of this component can have serious implications for the allowed motions of particles in such a field system, e.g., binding a particle to a given field-line equivalence class (FLEC) that is defined to be the class of all magnetic field lines that differ only by translation along the ignorable coordinate (or a rotation if the ignorable coordinate is an angle variable).

In the Appendix we show that if the electromagnetic *fields* do not depend on at least one coordinate, say x_2, in some orthogonal (not necessarily Cartesian) coordinate system, then a particular gauge can be found in which the vector and scalar potentials, A and ϕ, also do not depend on this coordinate. Furthermore, insofar as the component A_2 is concerned, this gauge is unique up to an overall constant in space and time. This implies that the Hamiltonian of a particle in such a field will have x_2 as an ignorable coordinate.

We will also show in § 2.1 that the scaled component of the vector potential, $h_2 A_2$ (a scale factor, h_i, converts a generalized orthogonal coordinate, x_i, into conventional length units such as centimeters), in the ignorable direction is a constant along a magnetic field line (as well as along the x_2 coordinate) and thus may be used as a label for a given FLEC where a FLEC is defined as all field lines that differ only in their x_2 coordinate, i.e., a surface generated by moving a field line through all values of the coordinate x_2. We will also show that a physically meaningful magnetic field-line motion may be defined. This motion reduces to the usual convection of field lines for ideal hydromagnetic systems and cannot be due to a simple gauge relabeling, since no remaining gauge transformations that preserve the symmetry can change the value of $h_2 A_2$. We will also show in § 2.2 that the physical scale of this binding is of the order of the harmonic mean of the particle's gyroradius in this field region.

2.1. *The Equivalence Class and Its Motion*

Since we have shown that with our choice of gauge nothing depends on the x_2 coordinate, we may establish an equivalence class of all field lines that differ only in their x_2 coordinate. Further, from the definition of B and the fact that $B \times B = B \times (\nabla \times A) = 0$, we may take the 2 component of $B \times B$ and obtain

$$B_1 \frac{1}{h_1} \frac{\partial(h_2 A_2)}{\partial x_1} + B_3 \frac{1}{h_3} \frac{\partial(h_2 A_2)}{\partial x_3} = B \cdot \nabla(h_2 A_2) = 0 , \quad (1)$$

and $h_2 A_2$ is constant along a magnetic field line. This means that we may *identify* a field line (or, rather, its equivalence class) by its value of $h_2 A_2$. Furthermore, if, at a later time, the FLEC with this value of $h_2 A_2$ is found at a different spatial position, we may *define* this motion as a field-line motion knowing that this motion is uniquely determined by the physics, because there is no remaining time-dependent gauge freedom that changes the value of A_2.

Employing the electric and magnetic fields, we define the velocity

$$w = \frac{c}{B_1^2 + B_3^2} E_2(\hat{e}_1 B_3 - \hat{e}_3 B_1) = -\frac{\partial(h_2 A_2)}{\partial t} \frac{\nabla(h_2 A_2)}{[\nabla(h_2 A_2)]^2} , \quad (2)$$

where $\hat{e}_1$ and $\hat{e}_3$ are unit vectors in the x_1 and x_3 directions, respectively. It is easy to see that w satisfies

$$\frac{\partial(h_2 A_2)}{\partial t} + w \cdot \nabla(h_2 A_2) = 0 \quad (3)$$

and may, therefore, be regarded as the velocity, perpendicular to the x_2 direction, of the FLECs, defined by their value of $h_2 A_2$. It is also clear that this velocity is identical to the $E \times B$ drift velocity whenever $E \cdot B = 0$.

2.2. *Binding the Particle to a Field Line (Equivalence Class)*

From this point the argument is nearly the same as JKG, and it is straightforward to see that a particle is "bound" in some sense to a particular value of $h_2 A_2$ and hence to a field line. Since x_2 is an ignorable coordinate, the x_2 component of the canonical momentum is conserved (Goldstein 1980), $h_2 p_2 + (e/c)h_2 A_2 = \text{const}$. Therefore, since $-p \le p_2 \le p$, we have

$$(h_2 A_2)_{\max} - (h_2 A_2)_{\min} \le \frac{2cp}{e} \bar{h}_2 , \quad (4)$$

where $\bar{h}_2 \equiv [(h_2)_{\max} + (h_2)_{\min}]/2$. In situations where the particles' energy, and hence p $[=(h_1^2 p_1^2 + h_2^2 p_2^2 + h_3^2 p_3^2)^{1/2}]$, does not change, this defines a flux tube in which the particle is constrained to remain.

To obtain an estimate of the spatial scale of this flux tube, consider a curve, S, in the x_1, x_3 plane, from the field line with the minimum value of $h_2 A_2$ to the one with the maximum value, which is everywhere perpendicular to the field $B'(x_1, x_3) \equiv \hat{e}_1 B_1(x_1, x_3) + \hat{e}_3 B_3(x_1, x_3)$, where $ds \cdot B' = 0$ along the curve S. We have then

$$(h_2 A_2)_{\max} - (h_2 A_2)_{\min} = \int_S dx_1 \frac{\partial(h_2 A_2)}{\partial x_1} + \int_S dx_3 \frac{\partial(h_2 A_2)}{\partial x_3}$$

$$= -\int_S (h_2 ds \times B')_2 \ge (h_2)_{\min} \int_S ds |B'| , \quad (5)$$

employing equation (A2). Since the absolute value of this integral $\le (2cp/e)\bar{h}_2$, we may divide equation (5) by this quantity to obtain

$$\int_S \frac{ds}{r_g'} \le \left[1 + \frac{(h_2)_{\max}}{(h_2)_{\min}}\right], \quad (6)$$

where $r_g' \equiv cp/eB'$ is the local gyroradius of a particle due to the x_1 and x_3 components of the magnetic field. A particle, therefore, moves a distance perpendicular to the x_1, x_3 component of the order of the harmonic mean of its gyroradius in this field along this path. Clearly, the electromagnetic fields may be arbitrary, which is consistent with Maxwell's equations and with reduced dimensionality.

It should be noted, however, that if B lies entirely in the x_2 direction over a region of space, $\partial(h_2 A_2)/\partial x_3$ and $\partial(h_2 A_2)/\partial x_1$ are zero, and $r_g' \to \infty$. There is, therefore, no restriction imposed by the theorem on particle motion in this region.

3. SPECIFIC CONSEQUENCES

The theorem established in the previous section makes it clear that a full discussion of any problem involving charged particles and electromagnetic fields either must be done in three dimensions or the effects of reduced dimensionality must be shown to be irrelevant to the problem at hand. This has important consequences for a number of

problems, including the physics of collisionless plasmas and cosmic-ray transport. We consider briefly some specific applications involving both physically real and artificially constrained symmetries.

3.1. *L Shell Conservation*

We will begin by considering a case in which an identifiable symmetry reflects (at least approximately) a *real* property of the system. It has been known for many years that the particles trapped in the geomagnetic field (the Van Allen belts) drift around the earth in such a way as to approximately conserve the "L" parameter. This parameter is essentially a label for the FLEC generated by the rotational symmetry of the dipole field. Since this symmetry is broken in many ways (the dipole is tilted with respect to the Earth's rotation axis and many perturbations of the field, both time dependent and static, do not preserve this symmetry), the conservation is only approximate but is good enough to be a useful concept in space physics. Furthermore, the entire concept of trapped (radiation belts) and excluded particles (cosmic-ray geomagnetic cutoff) was discussed by Störmer (1955) in terms of the "second first integral" of the motion, which is clearly the azimuthal component of the canonical momentum. In addition, the motion of the particles in the remaining degrees of freedom can often be understood by viewing the conserved momentum in the Hamiltonian as part of a potential function (Stern 1975). In such a case the actual symmetry of the system aids in the understanding of particle dynamics.

3.2. *Problems in Charged Particle Transport*

Previous discussions of cosmic-ray transport using the quasi-linear approximation (Jokipii 1966) often assumed irregularities that depended only on one coordinate (slab turbulence). In such geometries the field-line mixing or random walk was the *only* contribution to motion normal to the average magnetic field. Yet other studies of phenomenological particle scattering in a magnetic field (Forman, Jokipii, & Owens 1974; Jones, 1990) showed that cross-field diffusion should occur. It is now clear that the reduced dimensionality of the slab model, not the quasi-linear approximation, guaranteed this result and, since real turbulence is usually three dimensional in nature (Bieber, Wanner, & Matthaeus 1996), this limitation is probably not real, although the field-line mixing probably plays an important role.

Recently, Giacalone & Jokipii (1994) considered a number of issues in the general problem of cosmic-ray transport and its relation to the dimensionality of magnetic field fluctuations. Using synthesized magnetic field turbulence having a specific power spectrum, they directly verified that particles cannot cross field lines if the spectrum does not depend on all three spatial variables. However, when the turbulence was allowed to be fully three-dimensional, cross-field diffusion of the particles was evident. This has profound implications for simulations of particle acceleration by collisionless shocks.

3.3. *Collisionless Shocks*

As was discussed in the introduction, both hybrid and full particle one- and two-dimensional simulations do not predict any populations of ions accelerated to highly suprathermal energies in quasi-perpendicular shocks. This provides a major problem for these approaches: they are in direct conflict with ubiquitous observations of such populations near highly oblique traveling interplanetary shocks (e.g., Baring et al. 1997) and the general consensus that the anomalous cosmic-ray populations result from the acceleration of pick-up ions at the solar wind termination shock (e.g., Pesses, Jokipii, & Eichler 1981). The discussion of JKG pointed out that the important problem of charged-particle acceleration at quasi-perpendicular collisionless shocks could not be properly studied using one- or two-dimensional hybrid simulations. This is because cross-field motion plays a crucial role (Ellison, Baring, & Jones 1995). The restrictions for the acceleration at a purely perpendicular shock are particular severe. Here the magnetic field is carried downstream at the postshock flow speed. If the particles are to remain tied to a magnetic field line, as is required by the theorem, any acceleration at the shock would have to be fast enough for the particle gyroradius to increase as fast as the distance of the magnetic field line downstream. Otherwise, the particle would have to be convected downstream with the fluid, preventing any further acceleration. It is readily shown that the ordinary gradient drift along the shock face is too slow to effect the required increase in the gyroradius; however, see the discussion in § 3.4. Hence significant particle acceleration depends critically on the charged particles being able to scatter and diffuse across the magnetic field so that they can continue interacting with the shock. This necessary motion is artificially suppressed in simulations of reduced dimensions. Therefore, one would incorrectly infer that significant acceleration cannot occur in such quasi-perpendicular shocks. The situation in oblique shocks is not so severe, since particles may be able to follow field lines to continue interacting with the shock. But in such cases the quantitative character of the results will be affected because the cross-field transport is still important for efficient injection.

Furthermore, simulations of *quasi-parallel* shocks, where it has been noted that acceleration occurs by drifting along a transverse fluctuation in the magnetic field (Kucharek & Scholer 1991), should also be reexamined using three-dimensional simulations. In fact, Kucharek, Fujimoto, & Scholer (1993) demonstrated that significant changes are incurred in quasi-parallel shock simulations in going from one-dimensional to two-dimensional constructions.

3.4. *Acceleration by Field-Line Merging and Shock Surfing*

As examples of situations in which the conservation of the canonical momentum component does *not* prevent particle acceleration, we examine two cases where this conservation can actually *cause* the acceleration. Situations exist in which a charged particle is prevented from remaining near to the FLEC on which it was once found. This can arise because the FLEC ceases to exist (field-line merging) or a strong electric field temporarily prevents the particle from following the field-line motion (shock surfing).

For the first example, consider a planar current sheet in the y-z plane that separates two regions of oppositely directed magnetic field, $\boldsymbol{B} = \boldsymbol{e}_z B_0 \, \text{sgn}\,(x)$, where B_0 is a constant. We can choose $A_y(x) = B_0 |x|$ so field lines that are equidistant from the neutral sheet but on opposite sides (therefore of opposite sign) will have the same value of A_y. A given particle in the vicinity of this neutral sheet could range a farther distance in x than it could if the field did not change sign but must still maintain the constancy of the quantity $p_y + (q/c)A_y$. If now, as is shown in Figure 1, the
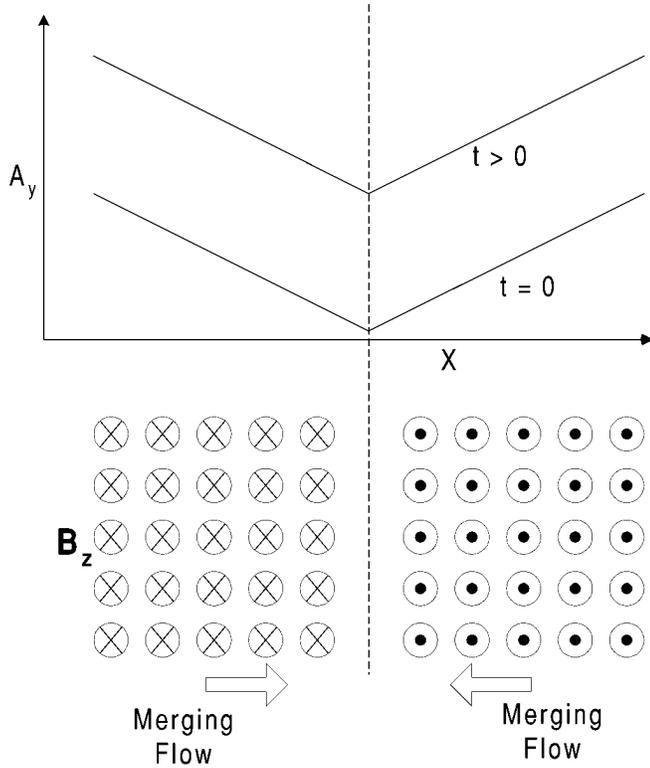
FIG. 1.—Motion of field lines in a region of field line merging (annihilation) showing the increase of the local value of A_y.

come this potential and thus cannot cross the shock because the sign of the potential drop is always in the direction to retard the incoming ions (Goodrich & Scudder 1984; Jones & Ellison 1987). Once again, if there is an ignorable direction perpendicular to the magnetic field, the particle must remain within a gyroradius or so of its original FLEC, which is proceeding through the shock front without the particle. The only way this can occur is for the particle's gyroradius to increase as fast as the field lines move downstream. This process, called shock surfing because the particle skims along the shock surface in the ignorable direction, continues until the particle gains sufficient energy to overcome the cross-shock potential.

Since both of these mechanisms depend rather strongly on there being no variation of the electric and magnetic fields in the ignorable direction, any perturbation that does not have that symmetry can "derail" the process and limit the energy gain to one considerably less than the theoretical limit based on that symmetry. Therefore, once again, theoretical investigations of these processes should allow for variations in three dimensions to determine the true limits on their effectiveness as accelerating mechanisms.

4. CONCLUSIONS

In this paper we have provided a rigorous proof of the JKG theorem and have emphasized that the most important application of the theorem is that the assumption of reduced dimensionality often used in modeling charged-particle motion has severe effects on the validity of the conclusions that may be drawn. In particular, any processes that may depend on particle motion across the magnetic field cannot, in principle, occur. This therefore casts doubt on those models or simulations that are complex enough that the contribution of cross-field motion cannot be independently assessed. In such cases the only recourse is to construct a fully three-dimensional model.

regions of oppositely directed fields are allowed to merge and annihilate field through plasma processes, which we do not discuss here, the value of A_y will everywhere continuously increase because of the constant loss of the smallest values in the vicinity of $x = 0$. In order to preserve the invariant value of $p_y + (q/c)A_y$, the value of $|p_y|$ must continuously increase. Thus p must continuously increase and the particle must be accelerated by the annihilating fields. Ambrosiano, Matthaeus, & Goldstein (1988) employ an argument similar to the above in a discussion of particle acceleration.

In a study of plasma shocks with a cross-shock electric potential drop, it has been shown (Zank et al. 1996; Lee, Shapiro, & Sagdeev 1996) that some particles incident on the shock front will not have enough kinetic energy to over-

APPENDIX

PROOF THAT A SUITABLE GAUGE EXISTS

Since there exist many symmetries other than linear translation for which electromagnetic fields are invariant, we consider an arbitrary, curvilinear, orthogonal coordinate system with coordinates x_1, x_2, and x_3. In general, such a coordinate system the line element is given by

$$ds = \hat{e}_1 h_1 \, dx_1 + \hat{e}_2 h_2 \, dx_2 + \hat{e}_3 h_3 \, dx_3 \,, \tag{A1}$$

where the $\hat{e}_i$ symbol is the unit vector in the ith direction and h_1, h_2, and h_3 are the length scale factors for the three coordinates (for a Cartesian coordinate system $h_1 = h_2 = h_3 = 1$). In this system we have

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{1}{h_1 h_2 h_3} \left\{ h_1 \hat{e}_1 \left[\frac{\partial(h_3 A_3)}{\partial x_2} - \frac{\partial(h_2 A_2)}{\partial x_3} \right] - h_2 \hat{e}_2 \left[\frac{\partial(h_3 A_3)}{\partial x_1} - \frac{\partial(h_1 A_1)}{\partial x_3} \right] + h_3 \hat{e}_3 \left[\frac{\partial(h_2 A_2)}{\partial x_1} - \frac{\partial(h_1 A_1)}{\partial x_2} \right] \right\} . \tag{A2}$$

Consider an electromagnetic field $\boldsymbol{E}(\boldsymbol{r}, t)$, $\boldsymbol{B}(\boldsymbol{r}, t)$, in which a particle of charge e and rest mass m_0 moves with velocity $\boldsymbol{v}$. The only restriction is that $\boldsymbol{E}$ and $\boldsymbol{B}$ are independent of one of the spatial coordinates, say, x_2.

We first demonstrate explicitly that if the electromagnetic field is not a function of a given coordinate, the vector potential $\boldsymbol{A}$ may be chosen to be independent of this coordinate as well. The first point to note is that this independence is manifested by the fact that the field structure may be moved an amount ds in the direction of this ignorable coordinate (if the coordinate is an angle variable the motion is a rotation), and the resulting field will be identical to the original. We emphasize that the transformations that follow are *not* coordinate transformations; they are translations or rotations of the fields, potentials, and gauge fields themselves. The transforms are *not* applied to Maxwell's equations; rather, Maxwell's equations are applied to the transformed fields. Therefore, an infinitesimal transformation gives

$$\begin{Bmatrix} \delta \boldsymbol{B} \\ \delta \boldsymbol{E} \end{Bmatrix} = ds\, \hat{e}_2 \cdot \nabla \begin{Bmatrix} \boldsymbol{B} \\ \boldsymbol{E} \end{Bmatrix} = \frac{ds}{h_2} \frac{\partial}{\partial x_2} \begin{Bmatrix} \boldsymbol{B} \\ \boldsymbol{E} \end{Bmatrix} = 0 \,,$$

$$\begin{Bmatrix} \delta \boldsymbol{A} \\ \delta \phi \end{Bmatrix} = ds\, \hat{e}_2 \cdot \nabla \begin{Bmatrix} \boldsymbol{A} \\ \phi \end{Bmatrix} = \frac{ds}{h_2} \frac{\partial}{\partial x_2} \begin{Bmatrix} \boldsymbol{A} \\ \phi \end{Bmatrix} \neq 0 \qquad \text{(in general)} \,, \tag{A3}$$

where $ds = |\boldsymbol{ds}|$. Since

$$\delta \boldsymbol{B} = \boldsymbol{B}' - \boldsymbol{B} = \nabla \times (\boldsymbol{A}' - \boldsymbol{A}) = \nabla \times (\delta \boldsymbol{A}') \,, \tag{A4}$$

$$\boldsymbol{A}' = \boldsymbol{A} + ds\, \frac{\partial \boldsymbol{A}}{h_2\, \partial x_2} \,, \tag{A5}$$

we have

$$\delta \boldsymbol{B} = ds\, \nabla \times \frac{1}{h_2} \frac{\partial \boldsymbol{A}}{\partial x_2} = 0 \,, \tag{A6}$$

so we may express the partial derivative with respect to x_2 as a potential field because its curl vanishes:

$$\frac{1}{h_2} \frac{\partial \boldsymbol{A}}{\partial x_2} = -\nabla \Psi \,, \tag{A7}$$

where Ψ is a scalar function. Since, similarly, for $\boldsymbol{E}$ we have

$$\delta \boldsymbol{E} = \boldsymbol{E}' - \boldsymbol{E} = -\nabla(\delta \phi) - \frac{1}{c} \frac{\partial}{\partial t}(\delta \boldsymbol{A}) = 0 \,, \tag{A8}$$

the condition on $\boldsymbol{E}$ is

$$\frac{1}{h_2} \frac{\partial \boldsymbol{E}}{\partial x_2} = -\nabla \frac{1}{h_2} \frac{\partial \phi}{\partial x_2} - \frac{1}{c} \frac{\partial}{\partial t} \frac{1}{h_2} \frac{\partial \boldsymbol{A}}{\partial x_2} \tag{A9}$$

$$= -\nabla \left(\frac{1}{h_2} \frac{\partial \phi}{\partial x_2} - \frac{1}{c} \frac{\partial \Psi}{\partial t} \right) = 0 \,. \tag{A10}$$

This requires

$$\frac{1}{h_2} \frac{\partial \phi}{\partial x_2} - \frac{1}{c} \frac{\partial \Psi}{\partial t} = F(t) \,, \tag{A11}$$

where $F(t)$ is a function of t alone.

We now may make a gauge transformation:

$$\boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} + \nabla \lambda \,, \qquad \phi \to \phi' = \phi - \frac{1}{c} \frac{\partial \lambda}{\partial t} \,, \tag{A12}$$

where λ is an arbitrary scalar function.

If we choose

$$\lambda = \int_C^{x_2} \Psi h_2\, dx'_2 + K(t) \int_C^{x_2} h_2\, dx'_2 \,, \tag{A13}$$

where $\int_C^{x_2}$ is an integral running from a fixed surface with $x_2 = \text{const}$ along a curve of constant x_1, x_3 and $K(t)$ is an arbitrary function of time alone, we see that

$$\frac{1}{h_2} \frac{\partial \boldsymbol{A}'}{\partial x_2} = \frac{1}{h_2} \frac{\partial \boldsymbol{A}}{\partial x_2} + \nabla \frac{1}{h_2} \frac{\partial \lambda}{\partial x_2} = \nabla \left(\frac{1}{h_2} \frac{\partial \lambda}{\partial x_2} - \Psi \right) = 0 \,, \tag{A14}$$

and we have gauged away any x_2 variation of $\boldsymbol{A}'$.

Similarly, we have, using equations (A11) and (A13),

$$\frac{1}{h_2}\frac{\partial \phi'}{\partial x_2} = \frac{1}{h_2}\frac{\partial \phi}{\partial x_2} - \frac{1}{c}\frac{\partial}{\partial t}\frac{1}{h_2}\frac{\partial \lambda}{\partial x_2} = \frac{1}{h_2}\frac{\partial \phi}{\partial x_2} - \frac{1}{c}\frac{\partial \Psi}{\partial t} - \frac{1}{c}\frac{\partial K(t)}{\partial t} = F(t) - \frac{1}{c}\frac{\partial K(t)}{\partial t} \ . \tag{A15}$$

So if we choose

$$K(t) = \int^t cF(t')dt' + K_0 \ , \tag{A16}$$

where K_0 is an arbitrary constant, we may also gauge away the x_2 variation in ϕ as well.

We can readily see that the value of A_2 is fixed, up to an overall constant in space and time, by the requirement that both A and ϕ are independent of the x_2 coordinate. For if one performs a further gauge transformation,

$$A \to A' = A + \nabla \Lambda \ , \tag{A17}$$

$$\phi \to \phi' = \phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t} \ , \tag{A18}$$

while requiring

$$\frac{1}{h_2}\frac{\partial A}{\partial x_2} = \frac{1}{h_2}\frac{\partial A'}{\partial x_2} = 0 \ ,$$

$$\frac{1}{h_2}\frac{\partial \phi}{\partial x_2} = \frac{1}{h_2}\frac{\partial \phi'}{\partial x_2} = 0 \ , \tag{A19}$$

we immediately obtain the requirement

$$\nabla\left(\frac{1}{h_2}\frac{\partial \Lambda}{\partial x_2}\right) = 0 \ , \qquad \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{h_2}\frac{\partial \Lambda}{\partial x_2}\right) = 0 \ , \tag{A20}$$

and $h_2^{-1}\partial\Lambda/\partial x_2 \, (=A_2' - A_2)$ is a constant in space and time, which is merely a new choice of K_0.

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